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Macroscopic Behavior of a Quantum Monomode Laser

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A kinetic equation for the density matrix of a monomode laser with explicit coupling with a thermal reservoir representing the cavity and a nonthermal one representing the pumping mechanism is derived. The macroscopic behavior of this system, inferred from Glauber's P function, is discussed within the framework of Glansdorff–Prigogine's theory of far-from-thermal-equilibrium open systems.

KEY WORDS: Reservoir-driven open systems; coherent states; entropy production; nonlinear equations; irreversible processes; stationary state; stability.

1. INTRODUCTION

The quantum theory of a laser oscillator is a fascinating problem in the sense that it deals with physical and mathematical concepts which are of primordial interest in contemporary physics.[1–3] Effectively this typical nonlinear system has to be described within the framework of nonequilibrium statistical mechanics as it consists of an open system in contact with two reservoirs, a thermal one—the cavity—and a nonthermal one—the lasing medium—and its distribution function, in the coherent representation, obeys a generalized Fokker–Planck equation. Moreover, as a far-from-thermal-equilibrium system, its macroscopic behavior deals with stability problems for the stationary states, and the possible emergence of a coherent state invites analogies with the appearance of a dissipative structure as well as a second-order phase transition.[4–6]

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The aim of this paper is to give a contribution to the study of the relation between a microscopic and a macroscopic description of this system.

We derive a kinetic equation for the density matrix of the monomode laser model studied in a previous paper (hereafter designated I)[7] but with an explicit pumping mechanism. For low concentration in active material we obtain, within the framework of the Prigogine–Résibois theory of non-equilibrium statistical mechanics,[8] a set of two coupled kinetic equations for the diagonal parts of the density matrix developed in a series of orthogonal operators. We recover, in the Markovian limit, Mandel's equation for a high-intensity laser. Since the macroscopic behavior of this system is described in Glauber's P function,[10] we may calculate, in the limit of weak fluctuations around macroscopic variables, the entropy production of this system. This quantity is found to be positive, and the stability of the macroscopic stationary state may be discussed within the framework of Glansdorff–Prigogine's stability criterion.[11]

In Section 2 the kinetic equation for the density matrix is derived, while approximate solutions are given in Section 3. The entropy production and the stability properties of the stationary state are discussed in Section 4, and formal or technical aspects have been postponed to the Appendix.

2. KINETIC EQUATIONS

We consider a monomode laser model described by a Hamiltonian which is an extension of the one studied in I. A pumping field is introduced explicitly since the pumping has to be a dynamical process rather than a static one. The Hamiltonian may then be written as follows (the notation is defined in I):

$$H = H_{\text{Dicke}} + H_{\text{Field cavity}} + H_{\text{pump}} \tag{1}$$

with

$$\begin{aligned}
H_{\text{Dicke}} &= \hbar\Omega \sum_{i=1}^{N} S_i^z + \hbar\omega a^+ a + \lambda \sum_{i=1}^{N} (S_i^+ a + S_i^- a^+) \\
H_{\text{Field cavity}} &= \sum_{\kappa} \hbar\omega_\kappa a_\kappa^+ a_\kappa + \sum_{\kappa} g_\kappa (a_\kappa^+ a + a_\kappa a^+) \\
H_{\text{pump}} &= \sum \hbar\omega_\alpha a_\alpha^+ a_\alpha + \sum_{i=1}^{N} \sum_{\alpha} [g_\alpha^- (S_i^+ a_\alpha + S_i^- a_\alpha^+) \\
&\quad + g_\alpha^+ (S_i^+ a_\alpha^+ + S_i^- a_\alpha)]
\end{aligned} \tag{2}$$

Moreover, we consider a situation where the direct spin–spin coupling and the coupling through the laser field are weak. This situation occurs in a laser with low concentration of active material where the probability of "re-collision" of atoms is negligible and corresponds to a mean field approximation.

With these approximations the density matrix may be factorized, and, since we are only interested in the properties of the atoms and of the laser mode, we trace this density matrix over all heat bath variables; consequently, $\rho(t) = \mathrm{Tr}'\,\Gamma(t)$ may be written as a series of orthogonal operators[12]

$$\rho(t) = \prod_{i=1}^{N} [\rho_i^{(0)}(t)I + \rho_i^{(z)}(t)S_i^z + \rho_i^{(+)}(t)S_i^+ + \rho_i^{(-)}(t)S_i^-] \tag{3}$$

Furthermore, since all spins are equivalent, $\rho_i^{(\varepsilon)} = \rho^{(\varepsilon)}$. We work in the diagonal representation for $a_\nu{}^+a_\nu$ and S_i^z and, in the corresponding $(\boldsymbol{\nu}, \mathbf{N})$ representation

$$\begin{aligned}[A_{\boldsymbol{\nu}}(\mathbf{N}) &= \langle \mathbf{N} + \tfrac{1}{2}\boldsymbol{\nu}|A|\mathbf{N} - \tfrac{1}{2}\boldsymbol{\nu}\rangle, |\mathbf{N}\rangle = \textstyle\prod_{i,\boldsymbol{\nu}} |m_i\rangle|N_{\boldsymbol{\nu}}\rangle, a^+a|N\rangle \\ &= N|N\rangle, S_i^z|m_i\rangle = m_i|m_i\rangle],\end{aligned}$$

$\rho(t)$ may formally be written as the iterative solution of the Liouville–von Neumann equation, which is, in the interaction picture

$$\tilde{\rho}_\nu(N, t) = \sum_{\nu'} \sum_{m=0} \mathrm{Tr}'\left\{\left(\frac{1}{i\hbar}\right)^n \int d\tau^n \langle \boldsymbol{\nu}|[\tilde{\mathscr{J}}(\tau)]^n|\boldsymbol{\nu}'\rangle \Gamma_{\boldsymbol{\nu}}(N, 0)\right\} \tag{4}$$

where the $\boldsymbol{\nu}$, $\mathbf{N}$ variables are those of the total system while ν, N belong to the proper laser system only. We have

$$\langle \nu|\tilde{\mathscr{J}}(\tau)|\nu'\rangle = \eta^{-\nu'}\tilde{V}_{\nu-\nu'}(N, \tau)\eta^{\nu} - \eta^{\nu'}\tilde{V}_{\nu-\nu'}(N, \tau)\eta^{-\nu} \tag{5}$$

with

$$\begin{aligned}\eta^\nu f(N) &= f(N + \tfrac{1}{2}\nu) \\ \tilde{V}(\tau) &= U(\tau)VU^+(\tau) \\ U(\tau) &= \exp i\Big[\Omega \sum_i S_i^z + \omega a^+a + \sum_\kappa \omega_\kappa a_\kappa{}^+ a_\kappa + \sum_\alpha \omega_\alpha a_\alpha{}^+ a_\alpha\Big]\tau \\ V &= \lambda \sum_{i=1}^{N} (S_i^+a + S_i^-a^+) + \sum_\kappa g_\kappa(a_\kappa{}^+a + a_\kappa a^+) \\ &\quad + \sum_{\alpha^*} \sum_{i=1}^{N} [g_\alpha{}^-(S_i^+a_\alpha + S_i^-a_\alpha{}^+) + g_\alpha{}^+(S_i^+a_\alpha{}^+ + S_i^-a_\alpha)]\end{aligned} \tag{6}$$

The initial density matrix is taken as

$$\Gamma(0) = \frac{1}{2^N} \sum_{i=1}^{N} (I + \alpha S_i^z) \prod_\kappa \rho_\kappa \prod_\alpha \rho_\alpha \rho \tag{7}$$

with $-2 \leqslant \alpha \leqslant 0$; ρ, ρ_κ, ρ_α are equilibrium field density matrices and, in order to simplify the calculations, they will be chosen as zero-photon-density matrices.

The kinetic equation for ρ may now be derived (see details in the Appendix) along the same lines as in I. Taking first the thermodynamic limit on the heat baths, we renormalize the spin propagators and the photon ones by their interactions with their respective heat bath, and, since these baths are taken to be singular, the contribution of the Laplace transform of a photon propagator is now

$$(z + \chi)^{-1}, \qquad \chi = \frac{2}{\pi\hbar^2}\sum_{\kappa} |g_\kappa|^2 \tag{8}$$

while the plus and minus spin propagators become, respectively,

$$(z + \gamma^+)[z(z + \gamma)]^{-1}, \qquad (z + \gamma^-)[z(z + \gamma)]^{-1} \tag{9}$$

with

$$\gamma^+ = \frac{2}{\pi\hbar^2}\sum_{\alpha} |g_\alpha{}^\pm|^2, \qquad \gamma = \gamma^+ + \gamma^-$$

The irreducibility condition is chosen to be the one associated with the vacuum $|0\rangle\langle 0|$, and the diagonal part of ρ obeys the integrodifferential equation[8]

$$\partial_t \rho_0^{(\epsilon)}(N, t) = \sum_{\epsilon'}\sum_{N'} \int d\tau\, G^{(\epsilon,\epsilon')}(N, N'|t - \tau)\rho_0^{(t')}(N', \tau), \qquad t, t' = 0, 2 \tag{10}$$

The nondiagonal parts are given by

$$\rho_\nu^{(\delta)}(N, t) = \sum_{\epsilon}\sum_{N'} \int d\tau\, C_{\nu 0}^{(\delta,\epsilon)}(N, N'|t - \tau)\rho_0^{(\epsilon)}(N', \tau) \tag{11}$$

As a consequence of the low-concentration approximation, these terms are small and will not be considered since we are concerned with photon and atom distributions. However, they may be important for some problems, e.g., frequency shifts, spin-field correlation functions, etc.

The kernels $G_0^{(\epsilon,\epsilon')}(N, N'|\tau)$ contain three parts:

(1) the explicit photon–cavity coupling (cf. I)

$$-\chi N\, \delta_{N,N'} + \chi(N + 1)\, \delta_{N+1,N'} \quad \text{in} \quad G_0^{(0,0)}$$

(2) the explicit pumping terms (cf. Appendix)

$$-\gamma\, \delta_{N,N'} \quad \text{in} \quad G_0^{(2,2)} \quad \text{and} \quad \gamma\eta\, \delta_{N,N'} \quad \text{in} \quad G_0^{(2,0)}$$

(3) the contributions due to the Dicke Hamiltonian itself with renormalized spin and photon propagators.

These contributions may be decomposed in a series of one-, two-,..., N-spin field interactions, respectively (cf. Fig. 1)

$$\psi = \sum_i \psi_i + \sum_{i \neq j} \psi_{i,j} + \cdots \tag{12}$$

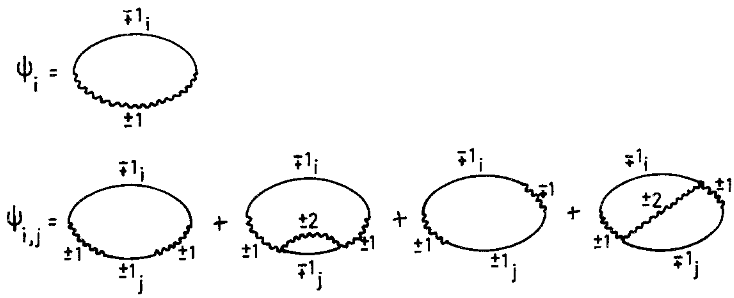

Fig. 1. Diagrammatic representation of the kernel of Eq. (10).

Since λ is proportional to $V^{-1/2}$, these interactions are proportional to powers of the spin concentration ($\sum_{i=1}^{N} \psi_i \sim N\lambda^2$, $\lambda \sim \lambda^* V^{-1/2}$, $\sum_{i=1}^{N} \psi_i \sim c\lambda^{*2}$, $\sum_{i,j=1}^{w} \psi_{i,j} \sim c^2\lambda^{*4}$).

For weak c we may consider only the one-spin term, this approximation being consistent with the factorization of the density matrix. The kinetic equations for the diagonal part of ρ may finally be written, in the Markovian limit,

$$\begin{aligned}
\partial_t \rho_0^{(0)}(N, t) &= -\chi[N - (N+1)\eta^2]\rho_0^{(0)}(N, t) \\
&\quad - q\{[2N + 1 - (N+1)\eta^2 - N\eta^{-2}]\rho_0^{(0)}(N, t) \\
&\quad + [1 + (N+1)\eta^2 - N\eta^{-2}]\rho_0^{(z)}(N, t)\} \\
\partial_t \rho_0^{(z)}(N, t) &= -\gamma\rho_0^{(z)}(N, t) + \gamma\eta\rho_0^{(0)}(N, t) \\
&\quad - q'\{(2N + 1 + (N+1)\eta^2 + N\eta^{-2})\rho_0^{(z)}(N, t) \\
&\quad + [1 - (N+1)\eta^2 + N\eta^{-2}]\rho_0^{(0)}(N, t)\}
\end{aligned} \tag{13}$$

In the Glauber–Sudarshan[10] representation defined by

$$\rho(t) = (1/\pi)\int d^2\alpha\, P(\alpha, \alpha^*, t)|\alpha\rangle\langle\alpha|,$$

$P^0(\alpha, \alpha^*, t)$ and $P^z(\alpha, \alpha^*, t)$ associated with $\rho_0^{(0)}(t)$ and $\rho_0^{(z)}(t)$ obey the following coupled Fokker–Planck equations:

$$\begin{aligned}
\partial_t P^0(\alpha, \alpha^*, t) &= \frac{1}{2}\left[\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right][\chi P^0(\alpha, \alpha^*, t) - qP^z(\alpha, \alpha^*, t)] \\
&\quad + q\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}[P^0(\alpha, \alpha^*, t) + P^z(\alpha, \alpha^*, t)] \\
\partial_t P^z(\alpha, \alpha^*, t) &= -\gamma[P^z(\alpha, \alpha^*, t) - \eta P^0(\alpha, \alpha^*, t)] \\
&\quad - q'\left\{2|\alpha|^2 P^z(\alpha, \alpha^*, t) - \left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*} - \frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right)\right. \\
&\quad \left. \times [P^0(\alpha, \alpha^*, t) + P^z(\alpha, \alpha^*, t)]\right\}
\end{aligned} \tag{14}$$

3. APPROXIMATE SOLUTIONS AND COMPARISON WITH OTHER THEORIES

In this section we discuss the relations between this theory and others and we present approximate solutions of Eq. (14) using physically meaningful assumptions. Note first that, since the characteristic time of the evolution of P^z is of order γ^{-1}, it varies slowly on the Markovian time scale and

$$P^z(\alpha, \alpha^*, t) \simeq P^z(\alpha, \alpha^*, 0)\exp[-(\gamma + 2q'|\alpha|^2)t] + \int_0^t d\tau \{\exp[-(\gamma + 2q'|\alpha|^2)(t - \tau)]\} \times \left[\gamma\eta + q|\alpha|^2 \frac{\partial}{\partial|\alpha|^2}\right] P^0(\alpha, \alpha^*, \tau) \tag{15}$$

so its long-time limit may be written

$$P^z(\alpha, \alpha^*, t) = s(\alpha, \alpha^*)P^0(\alpha, \alpha^*, t), \quad \langle s^z\rangle = \tfrac{1}{2}\int d^2\alpha\, sP^0$$

with

$$s(\alpha, \alpha^*) = (\gamma + 2q'|\alpha|^2)^{-1}\left[\gamma\eta + 2q'|\alpha|^2(P^0)^{-1}\frac{\partial}{\partial|\alpha|^2}P^0\right] \tag{16}$$

So the Markovian limit of (14) consists in the following Fokker–Planck equation:

$$\partial_t P^0(\alpha, \alpha^*, t) = \frac{1}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)[\chi - qs(\alpha, \alpha^*)] \times P^0(\alpha, \alpha^*, t) + q\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}[1 + s(\alpha, \alpha^*)P^0(\alpha, \alpha^*, t)] \tag{17}$$

The technique used to obtain the kinetic equation (14) may easily be shown to be equivalent to Haake's technique.[13] The new feature here is that the decomposition of the density matrix in a series of orthogonal operators and the mean-field approximation we made on it lead to an evolution operator which contains the complete one-atom contribution to all orders in the coupling constant. By this method we recover in fact in a direct and elegant way the equation derived recently by Mandel[9] after a complete resummation of the one-atom collision operator considered as a power series in the coupling parameter.

The time evolution of $P^0(\alpha, \alpha^*, t)$ may now be discussed through approximate solutions of Eq. (17). We shall only consider situations with $\eta \geqslant 0$, which are the only significant ones since for $\eta < 0$, $P^0(\alpha, \alpha^*, t)$ behaves like the initial chaotic distribution. [Note that for $\eta \simeq 0$ we have to take into account the $P^0(\alpha, \alpha^*, t)$ term of (16).]

Developing $P^0(\alpha, \alpha^*, t)$ as $\sum_k c_k(t)|\alpha|^2 \exp[-A(t)|\alpha|^2]$ in order to avoid convergence difficulties with the power series, and to put more light on the "distance" from the chaotic distribution, we obtain through (17) the following equations, in the long-time limit:

$$C_1(t) = \{A(t) - (\chi - q\eta)[q(1+\eta)]^{-1}\}C_0$$
$$C_2(t) = \left[(\chi - q\eta)^2 C_1(t) - \frac{4q^2\eta}{\gamma N} C_0\right](4q(1+\eta))^{-1} \tag{18}$$

while

$$\dot{A} = \dot{A}^< = A^<(\chi - q\eta) - q(1+\eta)A^{<2} \qquad \text{for} \quad \eta \ll \frac{\chi}{q} = \eta_c$$
$$\dot{A} = \dot{A}^0 = -\frac{2q^2\eta}{\gamma N} A^0 \qquad \text{for} \quad \eta \sim \eta_c \tag{19}$$
$$\dot{A} = \dot{A}^> = (\chi - q\eta) - q(1+\eta)A^> + \frac{2q^2\eta}{\gamma N} A^{>2} \qquad \text{for} \quad \eta \gg \eta_c$$

so that we have, respectively,

$$A^<(t) = (\chi - q\eta)\{q(1+\eta) + \alpha_0 \exp[-(\chi - q\eta)t]\}^{-1}$$
$$A^0(t) = \beta_0 \exp\left(-\frac{2q^2\eta}{\gamma N} t\right) \tag{20}$$
$$A^>(t) = (\chi - q\eta)[q(1+\eta)]^{-1} + \gamma_0 \exp[-q(1+\eta)t] \times \left[1 + \delta_0 \exp\left(\frac{4q^2}{\eta N}\frac{\chi - q\eta}{q(1+\eta)} t\right)\right]$$

$P^0(t)$ evolves to its stationary value with decay rates λ_i which are schematically represented in Fig. 2 and agree with other results[14] except that above threshold we do not recover the degeneracy of the λ_i.[12]

The stationary solution of (17) is

$$P_s^0(\alpha, \alpha^*) = Q^{-1} \frac{\gamma + 2q'|\alpha|^2}{\gamma(1+\eta) + 2q'|\alpha|^2} \times \left[\frac{\gamma(1+\eta) + 2q'|\alpha|^2}{\gamma(1+\eta)}\right]^{(\gamma/2q')\eta(1+\eta_c)} \exp(-\eta_c|\alpha|^2) \tag{21}$$

where Q is the normalization constant. The corresponding value of $s(\alpha, \alpha^*)$ is $\gamma\eta(\gamma + 2q'|\alpha|^2)^{-1}$. We see that for $\eta_c > 1$ the maximum of P^0 is given by $\bar{\alpha} = \bar{\alpha}^* = 0$, P^0 being roughly Gaussian, while for $\eta_c < 1$ and $\eta > \eta_c$, P^0 has its maximum at $|\bar{\alpha}|^2 = (\gamma N/2q)[(\eta/\eta_c) - 1]$. When the cavity losses are small and the pumping is strong, the peak is sharp and the system behaves nearly coherently.

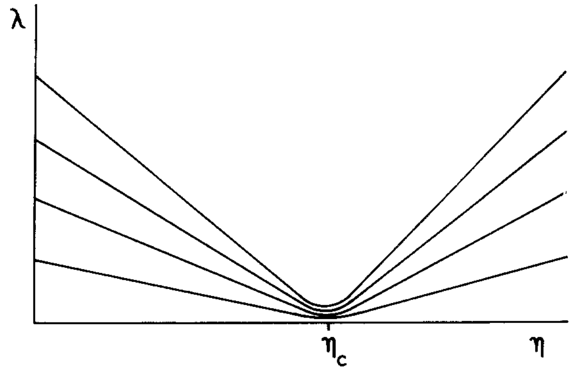

Fig. 2. Dependence of the decay rates of $P^0(\alpha, \alpha^*, t)$ vs. the pump parameter η (arbitrary scale).

As discussed by Mandel, the maximum of the stationary distribution (21) is, to vanishingly small correction $\gamma N(\eta - \eta_c)/2\chi$, above threshold and may differ strongly from Risken's[15] result: $\gamma N(\eta - \eta_c)/2q\eta$ in our notation.

These facts are among those which led to the comparison of this behavior with a second-order phase transition[4] and the expression of P^0 as the exponential of a thermodynamic-like potential obtained by developing P^0 in powers of α, α^*:

$$\begin{aligned} P^0(\alpha, \alpha^*) &= Q^{-1} \exp[-G(\alpha, \alpha^*)/\sigma] \\ G(\alpha, \alpha^*) &= [(\sigma_c - \sigma)|\alpha|^2 + \kappa|\alpha|^4 + \cdots] + G(0) \end{aligned} \tag{22}$$

with

$$\sigma = q(1 + \eta), \qquad \sigma_c = q(1 + \eta_c), \qquad \kappa = \frac{q^2\eta}{\gamma N}\frac{\sigma_c}{\sigma}$$

More interesting is the fact that the mean value of $|\alpha|^2$, which may be calculated from (21), consists of two parts; the first part, proportional to N, is precisely the value of $|\alpha|^2$ that maximizes P^0, while the other part, proportional to 1, may be neglected when N is great. So the separation between the classical and quantum effects, extensively discussed by Lieb and Hepp,[16] is recovered. Note that at threshold $\langle|\alpha|^2\rangle$ is proportional to $N^{1/2}$, as in all the theories where the asymptotic time limit $t \to \infty$ is taken prior to any eventual thermodynamic limit for the proper laser system itself. This leads to the macroscopic description of the monomode laser of Ginzburg–Landau type, as has been discussed elsewhere.[4,5] It is important to note that the fluctuations around the macrostate, which play an essential role in any realistic situation where N remains finite, are deduced from the microscopic theory.

The Ginzburg–Landau description of this system led to the comparison of the laser transition with a second-order phase transition, but with the creation of a dissipative structure. Let us briefly recall that[11] there exist systems which may have two different types of behavior:

A tendency to reach states of maximum disorder with destruction of structures. This behavior occurs mainly near the thermodynamic equilibrium.

A coherent behavior with creation of structures that may occur far from equilibrium. This ordering mechanism arises in open systems at a critical distance from equilibrium. The structures that are created in this situation are called dissipative structures.

Glansdorff and Prigogine have shown[11] that there is a general theory underlying the mechanism by which a system is driven to a new thermodynamic behavior beyond the instability of the thermodynamic one. This theory was developed for systems whose macroscopic description is based on the local equilibrium assumption.

We show in the following section in what sense laser action above threshold may be considered as a dissipative structure. Furthermore, the entropy production of this system is shown to obey the general law of the thermodynamics of irreversible processes in the absence of any local equilibrium assumption, which has obviously no sense in laser physics.

4. ENTROPY PRODUCTION AND STABILITY

The analogy of the laser transition to a second-order phase transition, the interpretation of the macroscopic state corresponding to laser action as a dissipative structure, and the relation between the macroscopic variables $\bar{\alpha}$, $\bar{\alpha}^*$ and the maximum of the Glauber–Sudarshan distribution function suggest the use of this distribution to discuss macroscopic aspects of this system, the mean value of an operator O being $\langle O\rangle = (1/\pi)\int d^2\alpha\, P^0\langle\alpha|O|\alpha\rangle$ (the macroscopic variable $\bar{O}$ is the value of $\langle\alpha|O|\alpha\rangle$ in the macroscopic state $\bar{\alpha}$, $\bar{\alpha}^*$ of the system).

The entropy of the laser field is defined as

$$S = -(k/\pi)\int d^2\alpha\, P^0(\alpha, \alpha^*, t)\langle\alpha|\ln \rho^0(t)|\alpha\rangle \tag{23}$$

where P^0 is the solution (17); the addition of high-intensity terms would only add small corrections but would not modify our conclusions. The entropy production may be written as

$$\mathscr{P} = \frac{d_iS}{dt} = \frac{2k}{\pi}\int d^2\alpha\, P^0(\alpha, \alpha^*, t)|(qs\alpha - q(P^0)^{-1}\,\partial_{\alpha^*}(1+s)P^0)\mathscr{F}(\alpha, \alpha^*, t)| \tag{24}$$

with

$$\mathscr{F}(\alpha, \alpha^*, t) = -\partial_\alpha\langle\alpha|\ \ln \rho^0(t)|\alpha\rangle \tag{25}$$

Since

$$\mathscr{F}(\alpha, \alpha^*, t) = \alpha^* \sum_N \frac{|\alpha|^{2N}\exp(-|\alpha|^2)}{N!} \times \ln\left[1 - \frac{\int d^2\beta\ |\beta|^{2(N+1)}(\exp-|\beta|^2)(\partial/\partial|\beta|^2)P^0(\beta, \beta^*, t)}{\int d^2\beta\ |\beta|^{2(N+1)}(\exp-|\beta|^2)P^0(\beta, \beta^*, t)}\right]$$

and due to the fact that $\int d^2\alpha\, P^0 = 1$, the entropy production is always positive. On the other hand, the macroscopic state of the system obeys the following evolution equations:

$$\begin{aligned} \dot{\bar{\alpha}} &= -(\chi - q\bar{s})\bar{\alpha} \\ \dot{\bar{s}} &= -\gamma(\bar{s} - \eta) - 2\rho'\bar{s}|\bar{\alpha}|^2 \end{aligned} \tag{26}$$

which are compatible with Lieb and Hepp's results.[16] We see that for $\eta_c > 1$ there exists only one stable stationary solution $\bar{\alpha} = 0$, $\bar{s} = \eta$ and that for $\eta_c < 1$ and $\eta > \eta_c$ an infinite set of stable solutions branch from $\bar{\alpha} = 0$, $\bar{s} = \eta$, namely,

$$\bar{\alpha} = \left[\frac{\gamma N}{2q}\left(\frac{\eta}{\eta_c} - 1\right)\right]^{1/2} e^{i\phi} \qquad (0 \leqslant \phi \leqslant 2\pi), \qquad \bar{s} = \eta_c$$

while the first one becomes unstable. The entropy associated with the macroscopic state $(\bar{\alpha}, \bar{\alpha}^*)$ of the laser system is $-k\langle\bar{\alpha}|\ \ln \rho^0|\bar{\alpha}\rangle$. While most of the concepts of irreversible thermodynamics are not applicable here, some interesting analogies with situations where a thermodynamic potential exists or where the local equilibrium assumption makes sense can be made. Since $|\bar{\alpha}|$ effectively plays the role of the order parameter of equilibrium phase transition and η is equivalent to the temperature, we see, for example, that at stationarity the macroscopic entropy $\bar{s}$ has a discontinuity in η at threshold as

$$\left.\frac{\partial \bar{s}}{\partial \eta}\right|_{\eta_c} = \begin{cases} -\dfrac{k}{I_0}\left.\dfrac{\partial I_0}{\partial \eta}\right|_{\eta_c} & \text{for} \quad \eta \underset{\to}{<} \eta_c \\ -\dfrac{k}{I_0}\left.\dfrac{\partial I_0}{\partial \eta}\right|_{\eta_c} - \left.\dfrac{k\gamma}{2\chi}\ln I_1\right|_{\eta_c} & \text{for} \quad \eta \underset{\to}{>} \eta_c \end{cases} \tag{27}$$

where

$$I_n = \int d^2\beta\ |\beta|^{2n}(\exp-|\beta|^2)P(\beta, \beta^*)$$

Moreover, the entropy production associated with the stable macroscopic state is, at stationarity,

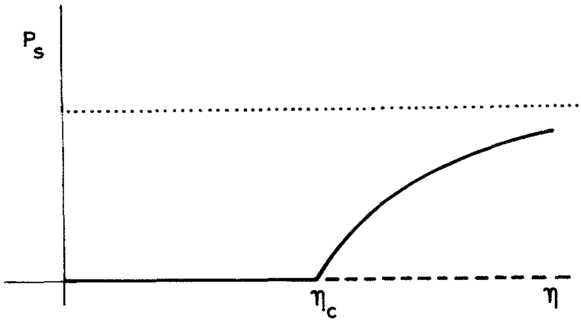

Fig. 3. Representation of the entropy production for the macroscopic stationary state vs. η.

$$\bar{\mathscr{P}}_s = \begin{cases} 0 & \text{for} \quad \bar{\alpha} = 0, \bar{s} = \eta \\ 2kq \dfrac{\eta_c^2}{1+\eta_c} \dfrac{\eta - \eta_c}{\eta} \left[1 + o\left(\dfrac{q\eta_c}{\gamma\eta N} \right) \right] & \\ \text{for} \quad |\bar{\alpha}|^2 = \dfrac{\gamma N}{2\chi}(\eta - \eta_c), \qquad \bar{s} = \eta_c & \end{cases} \tag{28}$$

Near threshold $\bar{\mathscr{P}}_s$ behaves like $\eta - \eta_c$, while far above threshold its behavior is nearly constant. $\partial \bar{\mathscr{P}}_s / \partial \eta$ shows a discontinuity at threshold which is represented in Figs. 3 and 4.

The macroscopic interpretation of (23) suggests that $\bar{\mathscr{P}}$ may be put into the thermodynamic form

$$\bar{\mathscr{P}} = J_{\bar{\alpha}} X_{\bar{\alpha}} + J_{\bar{\alpha}^*} X_{\bar{\alpha}^*} \tag{29}$$

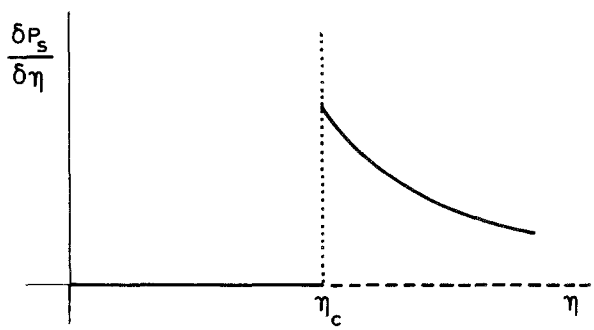

Fig. 4. Representation of the variation of entropy production for the macroscopic stationary state vs. η.

with

$$J_\alpha = qs\alpha - qP^{-1}\frac{\partial}{\partial\alpha^*}(1+s)P$$
$$X_\alpha = -k\frac{\partial}{\partial\alpha}\langle\alpha|\ln\rho|\alpha\rangle \tag{30}$$

$\langle\alpha|\ln\rho|\alpha\rangle$ plays the role of a pseudo-thermodynamic potential. The generalized force is zero for macroscopic states on the equilibrium branch ($\bar{\alpha} = 0$) and is nonzero on the other branch. Moreover,

$$\left|\frac{J_{\bar{\alpha}}}{X_{\bar{\alpha}}}\right|_s = \frac{2q\gamma\eta}{k}\frac{F(\bar{\alpha}, \bar{\alpha}^*)}{\gamma + 2\rho'|\alpha|^2}$$

It is interesting to note that the relation between $J_{\bar{\alpha}}$ and $X_{\bar{\alpha}}$ is linear only on the equilibrium branch and that the generalized Onsager coefficient $(2q\eta/k)F(0)$ becomes negative above threshold

$$\left(F^{-1}(0) = \ln\frac{\int d^2\beta\,(\exp-|\beta|^2)P}{\int d^2\beta\,|\beta|^2(\exp-|\beta|^2P}\right).$$

The variation of the entropy production due to these thermodynamic-like forces may be calculated in the macroscopic picture. It is zero at stationarity, and its dominant part may be written

$$\frac{d_X\bar{\mathscr{P}}}{dt} = -\frac{kq\bar{s}|\bar{\alpha}|^2}{1+\eta}[(\chi - q\bar{s})^2 + q\gamma\eta - q\bar{s}(\gamma + 2\rho'|\bar{\alpha}|^2)] \tag{31}$$

which is always negative since $\bar{s} \leqslant \gamma\eta(\gamma + 2\rho'|\bar{\alpha}|^2)^{-1}$.

Furthermore, the stability of the system in the vicinity of the stationary macroscopic state may be discussed as follows. It is easy to show, using (17)–(21), that

$$\overline{\delta s} = -k\bar{P}^{-1}\left[\frac{\overline{\partial P}}{\partial\bar{\alpha}}\bigg|_{\text{st}}\delta\bar{\alpha} + \frac{\overline{\partial P}}{\partial\bar{\alpha}^*}\bigg|_{\text{st}}\delta\bar{\alpha}^*\right]\langle\bar{\alpha}|\ln\rho|\bar{\alpha}\rangle = 0 \tag{32}$$

while

$$\overline{\delta^2 s} = -k\bar{P}^{-1}\left[\frac{\overline{\partial^2 P}}{\partial\bar{\alpha}^2}\bigg|_{\text{st}}(\delta\bar{\alpha})^2 + 2\frac{\overline{\partial^2 P}}{\partial\bar{\alpha}\,\partial\bar{\alpha}^*}\bigg|_{\text{st}}|\delta\bar{\alpha}|^2 + \frac{\overline{\partial^2 P}}{\partial\bar{\alpha}^{*2}}\bigg|_{\text{st}}(\delta\bar{\alpha}^*)^2\right]\langle\bar{\alpha}|\ln\rho|\bar{\alpha}\rangle \tag{33}$$

The laser system may be considered as an open system in contact with two heat baths: the cavity and the pump; the coupling with the cavity is constant, while the coupling with the pump varies. So if we subtract from $\overline{\delta^2 s}$ the effect of the variation of s due to the internal processes of the heat bath, namely, the effect corresponding to the $\gamma(s - \eta)$ term in $\dot{s}$, we obtain at stationarity:

$$\overline{\frac{\partial^2 G^s}{\partial\bar{\alpha}\,\partial\bar{\alpha}^*}}\bigg|_i = \frac{1}{1+\eta}\,\frac{\eta_c + (4\chi/\gamma)|\bar{\alpha}|^2 + [\eta_c/(1+\eta)]|\bar{\alpha}|^4}{(1 + a|\alpha|^2)^2} \geqslant 0 \tag{34}$$

and

$$\overline{\delta_i{}^2 s} = -\kappa|\delta\bar{\alpha}|^2 \qquad \text{with} \quad \kappa \geqslant 0 \tag{35}$$

Having subtracted the effect of the variation of the boundary conditions in $\overline{\delta^2 s_i}$, we see that the Glansdorff–Prigogine[11] criterion gives $\partial_t\, \delta_i{}^2 s|$ along the direction of motion as

$$\partial_t\, \overline{\delta_i{}^2 s}| = -\kappa[(\chi - q\bar{s})|\delta\bar{\alpha}|^2 - \tfrac{1}{2}q(\bar{\alpha}\,\delta\bar{\alpha}\,\delta\bar{s} + \bar{\alpha}^*\,\delta\bar{\alpha}\,\delta\bar{s})] \tag{36}$$

and around $\bar{\alpha} = 0$, $\bar{s} = \eta$ we have

$$\partial_t\, \overline{\delta_i{}^2 s} \geqslant 0 \qquad \text{for} \quad \eta \leqslant \eta_c$$

$$\partial_t\, \overline{\delta_i{}^2 s} < 0 \qquad \text{for} \quad \eta > \eta_c$$

while around

$$|\bar{\alpha}|^2 = \frac{\gamma N}{2q}\left(\frac{\eta}{\eta_c} - 1\right)$$

$\bar{s} = \eta_c$, we have

$$\partial_t\, \delta_i{}^2 s > 0 \tag{37}$$

We may conclude in complete accordance with previous results[6,7,16] that the state $\bar{\alpha} = 0$, $\bar{s} = \eta$ becomes unstable at threshold, where it bifurcates to an infinite set of stable states

$$\bar{\alpha} = \left[\frac{\gamma N}{2\chi}(\eta - \eta_c)\right]^{1/2} e^{i\phi}, \qquad \bar{s} = \eta_c$$

Another picture of this may be given in the following way: Since $P^0(\alpha, \alpha^*)$ is the probability for the system to be in the macroscopic state α, α^*, the evolution of this state being given by $\dot{\alpha} = -\partial G/\partial\alpha^*$, we may interpret $G(\alpha, \alpha^*)$ as a kinetic potential since from (31) it is easy to see that $\overline{d_X \mathscr{P}}$ and $\overline{dG}$ are proportional. Since G is defined through (17) and (21), we obtain $dG = 0$ at the stationary states, while $d^2G/d|\alpha|^2$ is proportional to $(\eta_c - \eta)/(1 + \eta)$ at ($\bar{\alpha} = 0$, $\bar{s} = \eta$) and $\eta_c(\eta - \eta_c)/\eta(1 + \eta_c)$ at

$$\{|\bar{\alpha}| = [\gamma N/2\chi)(\eta - \eta_c)]^{1/2} e^{i\phi}, \qquad \bar{s} = \eta_c\}.$$

So the condition $d^2G/d|\alpha|^2 \geqslant 0$ is verified for the stable stationary states.

Furthermore, the probability of small fluctuations around the macroscopic stationary state may be written, using (22) (for $k \ll 1$)

$$P(\delta\bar{\alpha}, \delta\bar{\alpha}^*) = \exp\left\{-\frac{1}{1+\eta}\left[\eta_c - \eta + \frac{4\chi}{\gamma N}|\bar{\alpha}|^2\right]|\delta\bar{\alpha}|^2\right\}$$

which on the equilibrium branch reduces to

$$\exp\left\{-\frac{\eta_c - \eta}{1+\eta}|\delta\bar{\alpha}|^2\right\}$$

and on the nonequilibrium branch to

$$\exp\left\{-\frac{1}{1+\eta}(\eta - \eta_c)|\delta\bar{\alpha}|^2\right\}$$

5. CONCLUSION

Having introduced explicitly the pumping as a dynamical process in the laser model we studied, and having developed the density matrix in a series of orthogonal spin operators, we were able to derive in the low-concentration approximation a set of two coupled kinetic equations for the density matrix of this system.

In the Markovian limit we recover Mandel's equation for a high-intensity laser, while the macroscopic description of the system is compatible with Lieb and Hepp's[16] results. Furthermore, this macroscopic description invites analogies with the thermodynamic description since quantities such as the entropy production may be defined and found to be positive. The change of this quantity due to thermodynamic-like forces is negative, while the stability of the system around its stationary macroscopic state may be discussed within the framework of Glansdorff–Prigogine's evolution criterion without any local equilibrium assumption.

It should be interesting now to study the fluctuations around the macroscopic variables as defined in (26) in order to gain a deeper insight into the relation between microscopic and macroscopic concepts in this problem and to study the emergence of the macroscopic state $\bar{\alpha}$, $\bar{\alpha}^*$. Moreover, a detailed knowledge of P^z, P^+, and P^- [cf. (10)–(11)] would also be interesting since they contain all the information we need to describe the spin's behavior in the laser process.

APPENDIX

The matrix elements $\langle\mu|\tilde{\mathscr{J}}(t)|\mu'\rangle$ appearing in (4) are represented by the same vertices as the ones defined in *I*. Their contributions are

$$
\begin{aligned}
\langle 0_i, \mu | S_i{}^+ a | -1_i, \mu + 1\rangle &= \eta^{-1_i}\delta_{M_i,0}(N+1) + \tfrac{1}{2}\mu)^{1/2}\eta^{1} \\
&\quad - \eta^{1_i}\delta_{M_i,0}(N - \tfrac{1}{2}\mu)^{1/2}\eta^{-1} \\
\langle 1_i, \mu | S_i{}^+ a | 0_i, \mu + 1\rangle &= \delta_{M_i,0}[\eta^{-1_i}(N+1+\tfrac{1}{2}\mu)^{1/2}\eta^{1} \\
&\quad - \eta^{1_i}(N - \tfrac{1}{2}\mu)^{1/2}\eta^{-1}] \\
\langle 0_i, \mu | S_i{}^- a^+ | 1_i, \mu - 1\rangle &= \eta^{1_i}\delta_{M_i,0}(N + \tfrac{1}{2}\mu)^{1/2}\eta^{-1} \\
&\quad - \eta^{-1_i}\delta_{M_i,0}(N + 1 - \tfrac{1}{2}\mu)^{1/2}\eta^{1} \\
\langle -1_i, \mu | S_i{}^- a^+ | 0_i, \mu - 1\rangle &= \delta_{M_i,0}[\eta^{1_i}(N+1+\tfrac{1}{2}\mu)^{1/2}\eta^{-1} \\
&\quad - \eta^{-1_i}(N - \tfrac{1}{2}\mu + 1)^{1/2}\eta^{1}] \\
\langle 0_i, \mu | S_i{}^+ a^+ | -1_i, \mu - 1\rangle &= \eta^{-1_i}\delta_{M_i,0}(N + \tfrac{1}{2}\mu)^{1/2}\eta^{-1} \\
&\quad - \eta^{1_i}\delta_{M_i,0}(N + 1 - \tfrac{1}{2}\mu)^{1/2}\eta^{1} \\
\langle 1_i, \mu | S_i{}^+ a^+ | 0_i, \mu - 1\rangle &= \delta_{M_i,0}[\eta^{-1_i}(N + \tfrac{1}{2}\mu)^{1/2}\eta^{-1} \\
&\quad - \eta^{1_i}(N + 1 - \tfrac{1}{2}\mu)^{1/2}\eta^{1}] \\
\langle 0_i, \mu | S_i{}^- a | 1_i, \mu + 1\rangle &= \eta^{1_i}\delta_{M_i,0}(N + 1 + \tfrac{1}{2}\mu)^{1/2}\eta^{1} \\
&\quad - \eta^{-1_i}\delta_{M_i,0}(N - \tfrac{1}{2}\mu)^{1/2}\eta^{-1} \\
\langle -1_i, \mu | S_i{}^- a | 0, \mu + 1\rangle &= \delta_{M_i,0}[\eta^{1_i}(N + 1 + \tfrac{1}{2}\mu)^{1/2}\eta^{1} \\
&\quad - \eta^{-1_i}(N - \tfrac{1}{2}\mu)^{1/2}\eta^{-1}] \\
\langle \mu, \mu_\kappa | a a_\kappa{}^+ | \mu + 1, \mu_\kappa - 1\rangle &= [(N + 1 + \tfrac{1}{2}\mu)(N + \tfrac{1}{2}\mu)_\kappa]^{1/2}\eta^{-1_\kappa}\eta^{1} \\
&\quad - [(N - \tfrac{1}{2}\mu)(N + 1 - \tfrac{1}{2}\mu)_\kappa]^{1/2}\eta^{1_\kappa}\eta^{-1} \\
\langle \mu, \mu_\kappa | a^+ a_\kappa | \mu - 1, \mu_\kappa + 1\rangle &= [(N + \tfrac{1}{2}\mu)(N + 1 + \tfrac{1}{2}\mu)_\kappa]^{1/2}\eta^{1_\kappa}\eta^{-1} \\
&\quad - [(N + 1 - \tfrac{1}{2}\mu)(N - \tfrac{1}{2}\mu)_\kappa]^{1/2}\eta^{-1_\kappa}\eta^{1} \qquad \text{(A.1)}
\end{aligned}
$$

The long-time divergences of each contribution of (4) are avoided by resumming the spin and field propagators by their respective heat-bath coupling. This has been done explicitly in I for the photon propagator, whose time dependence $F(N, t)$ obeys the following equation:

$$
\partial_t F(N, t) = \int_0^t d\tau \frac{2}{\hbar^2} \sum_\kappa |g_\kappa|^2 \cos[(\omega_\kappa - \omega)(t - \tau)]
$$

$$
\times \{N + \tfrac{1}{2} - [(N + \tfrac{1}{2})(N + \tfrac{3}{2})]^{1/2}\eta^2\} F(N, \tau) \qquad \text{(A.2)}
$$

Due to the singular character of the heat bath, the Laplace transform of this propagator may be written

$$
f(N, z) = (z + \chi)^{-1} + O(1/N) \qquad \text{(A.3)}
$$

with

$$
\chi = \frac{2}{\pi\hbar^2} \sum_\kappa |\rho_\kappa|^2
$$

Performing a Résibois–De Leener resummation[17] of the spin propagator, we obtain for the plus and minus spin propagators

$$
\begin{aligned}
f^{\pm}(z) = {} & \left[z + \frac{1}{\pi\hbar^2}\left(\sum_\alpha |\rho_\alpha{}^{\pm}|^2\langle N+1\rangle_\alpha + \sum_\alpha |g_\alpha{}^{\mp}|^2\langle N\rangle_\alpha\right)\right] \\
& \times \left[z + z\frac{1}{\pi\hbar^2}\sum_\alpha \langle 2N+1\rangle_\alpha(|\rho_\alpha{}^+|^2 + |\rho_\alpha{}^-|^2)\right. \\
& \left. + \frac{1}{\pi^2\hbar^4}\sum_\alpha \langle N(N+1)\rangle_\alpha(|\rho_\alpha{}^+|^2 - |\rho_\alpha{}^-|^2)^2\right]^{-1}
\end{aligned}
\tag{A.4}
$$

or, for zero-photon heat baths

$$
f^{\pm}(z) = (z + \gamma^{\pm})/z(z+\gamma) \tag{A.5}
$$

Taking now the irreducibility condition associated with the vacuum $|0\rangle\langle 0|$, we find that the diagonal part of ρ obeys the following equation:

$$
\partial_t \rho_0^{(\epsilon)}(t) = \sum_{\epsilon'} \int_0^t d\tau\, G_0^{(\epsilon,\epsilon')}(t-\tau)\rho_0^{(\epsilon')}(\tau), \qquad \epsilon, \epsilon' = 0, z \tag{A.6}
$$

$G_0^{(\epsilon,\epsilon')}$ contains the following contributions:

1. The explicit photon–cavity coupling as calculated in I and appearing in

$$
-\chi N + \chi(N+1)\eta^2 \tag{A.7}
$$

2. The explicit spin-pumping contribution is in the singular heat-bath approximation and in the thermodynamic limit

$$
\begin{aligned}
-\frac{1}{\pi\hbar^2}\sum_{N_\alpha}\sum_\alpha \{ & |g_\alpha{}^+|^2[\eta^{-1_i}\delta_{M_i,0}((N+1)_\alpha)^{1/2}\eta^{1_\alpha} \\
& - \eta^{1_i}\delta_{M_i,0}N_\alpha^{1/2}\eta^{-1_\alpha}]\delta_{M_i,0}[\eta^{1_i}((N+\tfrac{1}{2})_\alpha)^{1/2}\eta^{-1_\alpha} \\
& - \eta^{-1_i}((N+\tfrac{1}{2})_\alpha)^{1/2}\eta^{1_\alpha}]\rho(N_\alpha) + |g_\alpha{}^-|^2[\eta^{-1_i}\delta_{M_i,0} \\
& \times N_\alpha^{1/2}\eta^{-1_\alpha}\eta^{1_i}\delta_{M_i,0}((N+1)_\alpha)^{1/2}\eta^{1_\alpha}]\delta_{M_i,0}[\eta^{1_i} \\
& \times ((N+\tfrac{1}{2})_\alpha)^{1/2}\eta^{1_\alpha} - \eta^{-1_i}((N+\tfrac{1}{2})_\alpha)^{1/2}\eta^{-1_\alpha}]\rho(N_\alpha)\}
\end{aligned}
\tag{A.8}
$$

leading, for zero-photon baths, to

$$
\begin{aligned}
-\frac{1}{\pi\hbar^2}\Bigg[& (\delta_{M_i-\frac{1}{2},0} - \delta_{M_i+\frac{1}{2},0}\eta^{2_i})\sum_\alpha |\rho_\alpha{}^+|^2 \\
& - (\delta_{M_i-\frac{1}{2},0}\eta^{-2_i} - \delta_{M_i+\frac{1}{2},0})\sum_\alpha |\rho_\alpha{}^-|^2\Bigg]
\end{aligned}
\tag{A.9}
$$

or $-\gamma$ in $G_0^{(z,z)}$ and $\gamma\eta$ in $G_0^{(z,0)}$ with

$$
\gamma = \frac{2}{\pi\hbar^2}\sum_\alpha (|g_\alpha{}^+|^2 + |\rho_\alpha{}^-|^2)
$$

and

$$\eta = \frac{\sum_\alpha (|g_\alpha{}^+|^2 - |\rho_\alpha{}^-|^2)}{\sum_\alpha (|\rho_\alpha{}^+|^2 + |g_\alpha{}^-|^2)}$$

The remaining graphs appearing in the kernel are of one- or multiple-spin type. Considering only the on-spin terms, this approximation, valid for low concentrations of active material, is consistent with the factorization of the density matrix—the kernel is the sum of the following contributions:

in	$G_0^{(0,0)}$:	$R(z)[(2N + 1) - (N + 1)\eta^2 - N\eta^{-2}]$
in	$G_0^{(0,\bar{z})}$:	$R(z)[1 + (N + 1)\eta^2 - N\eta^{-2}]$
in	$G_0^{(z,0)}$:	$R(z)[1 - (N + 1)\eta^2 + N\eta^{-2}]$
in	$G_0^{(z,\bar{z})}$:	$R(z)[(2N + 1) + (N + 1)\eta^2 + N\eta^{-2}]$

where

$$R(z) = F^+(z + i(\Omega - \omega)) + F^-(z - i(\Omega - \omega))$$

with

$$F^+(z) = (1/2\pi) \oint dz' f(z - z')f^+(z')$$

In the Markovian limit of (A.6) (according to the short memory of the heat baths we may safely assume that this limit exists) only $R(0)$ appears and is calculated to be

$$q = R(0) = \frac{2N\lambda^2(\chi + \gamma)^2}{\Delta^2 + (\chi + \gamma)^2} f(\chi, \gamma) = Nq' \tag{A.10}$$

with $\Delta = \Omega - \omega$ and $f(\chi, \gamma) \simeq 2\chi^{-1}$ for $\gamma < \chi$, or χ^{-1} for $\gamma > \chi$. The case $\gamma < \chi$ corresponds to a gas laser, while the case $\gamma > \chi$ occurs for a solid-state laser, so that q(gas laser) $\simeq 2q$(solid-state laser).

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